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Average stress in a Stokes suspension of disks

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Abstract

The ensemble-average velocity and pressure in an unbounded quasi-random suspension of disks (or aligned cylinders) are calculated in terms of average multipoles allowing for the possibility of spatial non-uniformities in the system. An expression for the stress due to the suspended particles is deduced from these results. It is found that spatial non-uniformity can induce an antisymmetric component in this stress even when no external couple acts on the particles. This component has the same order of magnitude as the term responsible for the difference between the effective viscosity of the suspension and that of the pure fluid. General considerations and a simple cell model suggest that the antisymmetric component will induce a flow in the presence of gradients of the particle volume fraction or of the relative interphase velocity, for example in a sedimenting suspension with a horizontally non-uniform particle distribution. While the derivation assumes Stokes flow conditions for the local flow around the particles, the Reynolds number of the mean macroscopic flow is unrestricted. In addition to illustrating the general nature of the particle stress, this work is a necessary prerequisite for the development of a closed suspension model on the basis of direct numerical simulations.

Keywords: Suspensions; Averaged equations; Constitutive relations; Stokes flow; Multi-body interactions

1. Introduction

The derivation of a reliable averaged-equations model for disperse multiphase flow is an important problem which, in spite of a significant research effort over several decades, is still far

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from a satisfactory solution. The chief and well-known difficulty is the closure of the equations derived from the formal averaging of the exact microscopic formulation: the simplest closure assumptions lead to models which turn out to be lacking not only in physical realism, but in mathematical behavior as well.

One of the most promising ways to make progress toward this goal is to use direct numerical simulations of the exact microscopic formulation to gain the insight necessary to close the equations. The literature contains many instances of this approach. For example, Ladd (1990) (see also Ladd and Verberg, 2001) and others (e.g. Brady and Bossis, 1985; Brady et al., 1988; Chang and Powell, 1993; Mo and Sangani, 1994; Sangani and Mo, 1996) have calculated the effective viscosity of suspensions of spheres, Spelt and Sangani (1997/1998) and Kang et al. (1997) have derived models for bubbly liquids, Tsao and Koch (1995) and Sangani et al. (1996) have studied gas–solid suspensions, Sundararajakumar et al. (1994) have calculated the effective transport properties of dispersions of disks, Martys et al. (1994) have studied the effective viscosity of Brinkman's equation, Higdon and Ford (1996) and Clague and Phillips (1997) have calculated the permeability of fibrous media. In all of these studies, as well as in many others which we do not cite for brevity, the system is assumed to be essentially spatially uniform.

While these results represent a significant progress in our understanding of several facets of disperse multiphase flows, they are not sufficient to construct a theory of sufficient generality to be used in the presence of the significant spatial non-uniformities which naturally arise in disperse systems, such as bubbles in fluidized beds, sedimenting fronts, kinematic shocks, high-vorticity regions, and others. For example, in a homogeneous viscous suspension subject to simple shear, the average particle velocity $\overline{\mathbf{w}}$ is identical to the mixture velocity $\mathbf{u}_{\mathrm{m}}$, and therefore contributions to the stress induced by a gradient of the slip velocity $\overline{\mathbf{w}} - \mathbf{u}_{\mathrm{m}}$, which would arise in more general flows, cannot be identified. In a homogeneous settling suspension all gradients vanish and, therefore, contributions to the interphase force analogous to the Faxèn force cannot be found. Many similar examples could be cited: the point is that, for a computational approach to closure to be fully successful, the closure problem itself must be posed in a general context and, in particular, avoiding the assumption of spatial uniformity.

These considerations have motivated our work on the proper definition of the mixture pressure and the general structure of the average stress in disperse flows (Marchioro et al., 1999; Prosperetti, 2003), and on the averaged description of Stokes suspensions of spheres (Zhang and Prosperetti, 1997; Marchioro et al., 2000, 2001; Wang and Prosperetti, 2001; Tanksley and Prosperetti, 2001). Among other results, we found that, in general, the average stress tensor ceases to be symmetric even in the absence of couples acting on the particles, an adequate description of the symmetric part of the stress tensor requires the introduction of other effective viscosities in addition to the well-known Einstein one, the average interphase force cannot be reduced to drag but contains several contributions, etc. In Tanksley and Prosperetti (2001) we have presented an exact, if implicit, direct calculation of the average stress in a suspension of spheres which clearly exhibits the structure of the stress tensor and forms a useful basis for both analytical and numerical attempts at a closure of the averaged equations.

The purpose of this paper is to present a similar calculation for the two-dimensional case of a suspension of disks (or aligned cylinders). The motivations are several. In the first place, the technical details are somewhat simpler to follow, which renders the conclusion more transparent. Secondly, by pointing out that essentially all the features encountered in the three-dimensional case are also present here, one may better appreciate their universality and, at least qualitatively, check the earlier calculation. Thirdly, the task of numerically deducing closure relations, that we have begun in earlier papers (Marchioro et al., 2000, 2001; Wang and Prosperetti, 2001), is easier in two dimensions and can therefore be brought to a more satisfactory degree of completion: the results derived here are a necessary prerequisite for this latter task. In spite of the somewhat limited physical relevance of the two-dimensional problem, this is an important point as it may be expected that the general solution of the closure problem will be facilitated by the existence of instances of specific cases in which accurate results are available. Finally, the present results also provide a further explicit check of the definition of mixture pressure derived by formal means in Marchioro et al. (1999).

In the limit of a spatially uniform suspension, with due account for the difference in space dimensionality, our results agree with those presented in the well-known study by Batchelor (1970). While the literature on suspensions is vast, very few papers address the situation of spatial non-uniformity considered here. Feuillebois (1984) studied the case of vertical inhomogeneities, but he was interested in deriving explicit corrections to the settling velocity for specific cases, rather than formulating a general theory. While here we consider both phases, the work of Lhuillier (1992) and Lhuillier and Nozières (1992) focused on the disperse phase and only considered weak inhomogeneities. In Marchioro and Prosperetti (1999) we studied heat conduction in a non-uniform composite.

The dominant contributions to the new effects that we identify are of the same order as those responsible for the difference between the effective viscosity of the suspension and the viscosity of the pure fluid. Higher-order corrections are of the order of the ratio of the particle radius a, or the mean interparticle distance $a/\beta_D^{1/2}$ (with β_D the particle volume fraction), to the macroscopic length scale L. These higher-order terms may be considered as embodying non-local corrections caused by particle–particle interactions and by the finite size of the particles. The latter effects are similar in nature to the Faxén term in the force on a sphere. While small for a uniform system, all these terms may become appreciable in the presence of non-zero gradients, such as near a sedimenting front, at the edge of the "clumps" of the suspended phase produced by inertial effects or convective currents (see e.g. Squires and Eaton, 1991; Wang and Maxey, 1993), and others. Furthermore, these terms contain spatial derivatives and, therefore, affect the short-scale behavior (in particular, well-posedness and continuous dependence on the data) upon which the mathematical structure of the equations depends.

As a final point, it may be noted that the results that follow are also applicable to a two-dimensional porous medium, rather than a suspension, provided the particle velocity is set to zero. This problem, which physically corresponds to flow through a bed of aligned fibers, has also been studied in the past, notably by Howells (1974, 1998), Shaqfeh and Fredrickson (1990), Ghaddar (1995), James and Davis (2001), and others.

The present derivation is based on a general solution of the two-dimensional Stokes equations (given in Appendix A) patterned after the well-known Lamb solution for spheres. Several other details of the calculation are given in Appendices A–D and, in a much expanded version, in a document available from the author.

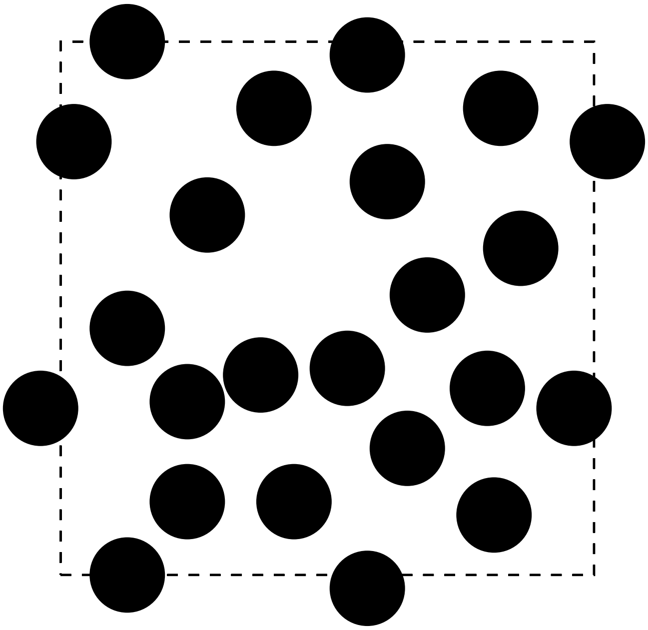

Fig. 1. A fundamental cell in a quasi-random suspension of disks.

2. Ensemble averages for a non-uniform suspension

We consider an approximation to an unbounded two-dimensional suspension (or porous medium) of disks (or aligned cylinders) consisting of the periodic repetition of a square cell of side L and volume (area) $\mathscr{V}$, into which N identical disks are randomly placed (see Fig. 1). [2] Neither the parameter L nor the particle number N appear explicitly in the final result and (provided the required averages converge) can then be taken arbitrarily large.

Let $\mathbf{y}^\alpha$, $\alpha = 1, 2, \ldots, N$, be the coordinates of the disk centers in the fundamental cell distributed according to a probability density $P(N) \equiv P(\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N)$ normalized so that

$$\int \mathrm{d}\mathscr{C}^N P(N) = N!, \tag{1}$$

in view of the identity of the disks. Here $\mathrm{d}\mathscr{C}^N = \mathrm{d}^2y^1\,\mathrm{d}^2y^2\cdots\mathrm{d}^2y^N$, and, for each variable, the integration ranges over the entire cell $\mathscr{V}$. In general, P will also depend on time, but the dependence on this variable is non-essential and is omitted throughout the present paper.

For each configuration $\mathscr{C}^N \equiv \{\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^N\}$, the indicator function for the region occupied by the disks is

[2] If the Fourier expansions that follow are generalized to include wave numbers of the reciprocal lattice, the present results are also applicable to a fundamental cell in the shape of a parallelogram. This extension is not of great interest here as the specific nature of the cell disappears from the final results.

$$\chi(\mathbf{x};N)=\sum_{\alpha=1}^{N}H(a-|\mathbf{x}-\mathbf{y}^{\alpha}|), \tag{2}$$

where H is the Heaviside distribution, and a the common radius of the disks. The probability that a point $\mathbf{x}$ be in the continuous phase (index C) is the continuous-phase volume fraction and is given by

$$\beta_{\mathrm{C}}(\mathbf{x})=\frac{1}{N!}\int \mathrm{d}\mathscr{C}^{N}P(N)[1-\chi(\mathbf{x};N)]; \tag{3}$$

the probability for the point $\mathbf{x}$ to be in the disperse phase (index D) is $\beta_{\mathrm{D}}=1-\beta_{\mathrm{C}}$.

With the assumption of inertialess (Stokes) flow, given a deterministic forcing agent, such as a force applied to the particles, or an imposed shear, the behavior of each realization of the ensemble is entirely determined by the instantaneous position of the particle centers. Thus, the phase-ensemble average for the generic continuous-phase field f (such as pressure, velocity, etc.) may be written as

$$\beta_{\mathrm{C}}(\mathbf{x})\langle f\rangle(\mathbf{x})=\frac{1}{N!}\int \mathrm{d}\mathscr{C}^{N}P(N)(1-\chi)f(\mathbf{x};N). \tag{4}$$

If the field f is spatially periodic, its phase-ensemble average can be expanded in a Fourier series:

$$\beta_{\mathrm{C}}(\mathbf{x})\langle f\rangle(\mathbf{x})=f_0+\sum_{\mathbf{k}\neq 0}f_{\mathbf{k}}\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}), \tag{5}$$

where the summation is over all wave numbers that are compatible with the dimensions of the cell (excluding $\mathbf{k}=0$, which is treated separately), and

$$f_0=\frac{1}{\mathscr{V}}\int \mathrm{d}^2x\beta_{\mathrm{C}}(\mathbf{x})\langle f\rangle(\mathbf{x}),\quad f_{\mathbf{k}}=\frac{1}{\mathscr{V}}\int \mathrm{d}^2x\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\beta_{\mathrm{C}}(\mathbf{x})\langle f\rangle(\mathbf{x}), \tag{6}$$

or, from (4),

$$f_0=\frac{1}{N!}\int \mathrm{d}\mathscr{C}^{N}P(N)\left[\frac{1}{\mathscr{V}}\int_{\mathscr{L}}\mathrm{d}^2xf(\mathbf{x};N)\right], \tag{7}$$

$$f_{\mathbf{k}}=\frac{1}{N!}\int \mathrm{d}\mathscr{C}^{N}P(N)\left[\frac{1}{\mathscr{V}}\int_{\mathscr{L}}\mathrm{d}^2x\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})f(\mathbf{x};N)\right], \tag{8}$$

where, due to the presence of the factor $(1-\chi)$ in the definition (4) of phase ensemble average, the $\mathbf{x}$-integration ranges only over the portion $\mathscr{L}$ of the fundamental cell occupied by the continuous phase. Thus, the calculation of the Fourier coefficients f_0 and $f_{\mathbf{k}}$ requires the evaluation of integrals of the form

$$F_0(N)=\int_{\mathscr{L}}\mathrm{d}^2xf(\mathbf{x};N), \tag{9}$$

$$F_{\mathbf{k}}(N)=\int_{\mathscr{L}}\mathrm{d}^2x\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})f(\mathbf{x};N). \tag{10}$$

For integrals of the second type we use the identity $\nabla^2\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})=-k^2\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})$ and Green's theorem to write

$$F_{\mathbf{k}}(N) = \frac{1}{k^2}\int_{\partial\mathscr{L}} \mathrm{d}A\mathbf{n}\cdot[\mathrm{i}\mathbf{k}f - \mathbf{\nabla}f]\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}) - \frac{1}{k^2}\int_{\mathscr{L}} \mathrm{d}^2x\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\nabla^2 f, \tag{11}$$

where the integration in the first term is over the entire boundary of the continuous phase (i.e., the particle surface and the cell boundary) and the unit normal $\mathbf{n}$ is directed into the integration domain (and hence out of the particles on the particle surfaces). The cell boundary actually gives a vanishing contribution because of periodicity and this term reduces then to the sum of integrals over the surface (perimeter) of the disks. Thus, upon setting $\mathbf{x} = \mathbf{y}^\alpha + \mathbf{r}$ in the integral over the surface of the particle α, we may write

$$F_{\mathbf{k}}(N) = \sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha)L_{\mathbf{k}}^\alpha - \frac{1}{k^2}\int_{\mathscr{L}} \mathrm{d}^2x\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\nabla^2 f, \tag{12}$$

where

$$L_{\mathbf{k}}^\alpha = \frac{1}{k^2}\int_{r=a} \mathrm{d}S^\alpha\mathbf{n}\cdot[\mathrm{i}\mathbf{k}f(\mathbf{y}^\alpha+\mathbf{r}) - \mathbf{\nabla}_r f(\mathbf{y}^\alpha+\mathbf{r})]\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}). \tag{13}$$

The last term in (12) vanishes in the case of the pressure, which is harmonic and could be handled by a repeated application of the same identity in the case of the velocity, which is biharmonic, although, as shown in Section 5, a different approach is more convenient in this case. Integrals of the form (9) can be treated similarly finding

$$F_0(N) = \sum_{\alpha=1}^{N} L_0^\alpha, \tag{14}$$

where

$$L_0^\alpha = -\int_{r=a} \mathrm{d}S^\alpha\mathbf{n}\cdot\mathbf{\nabla}_r f_*(\mathbf{y}^\alpha+\mathbf{r}). \tag{15}$$

in which f_* is a particular solution of

$$\nabla^2 f_* = f. \tag{16}$$

3. Microscopic velocity and pressure fields

We take the continuous phase to be an incompressible fluid and write its velocity in the form

$$\mathbf{u}(\mathbf{x};N) = \mathbf{U}_\infty(\mathbf{x}) + \mathbf{v}(\mathbf{x};N), \tag{17}$$

where the first term is a deterministic imposed velocity and $\mathbf{v}$, the disturbance due to the particles, is periodic; for the situations of concern here, it is sufficient to take $\mathbf{U}_\infty$ in the form

$$\mathbf{U}_\infty(\mathbf{x}) = \mathbf{U}_0 + \boldsymbol{\gamma}\cdot\mathbf{x}, \tag{18}$$

in which $\mathbf{U}_0$ is a constant vector and $\boldsymbol{\gamma}$ is a constant traceless two-tensor. Similarly, we write the pressure field as

$$p(\mathbf{x};N) = P_\infty(\mathbf{x};N) + \mu q(\mathbf{x};N), \tag{19}$$

where μ is the continuous-phase viscosity, ∇P_∞ is a constant, and q is periodic. If $\mathbf{G}$ is the (constant) body force acting on the fluid, the Stokes equation is

$$\nabla \cdot (-p\mathbf{I} + \boldsymbol{\tau}) + \mathbf{G} = 0, \tag{20}$$

in which $\mathbf{I}$ is the identity two-tensor and $\boldsymbol{\tau}$ the viscous stress. When this equation is integrated over the continuous phase in the fundamental cell one finds, by the divergence theorem,

$$(\mathscr{V} - Nv)(-\nabla P_\infty + \mathbf{G}) - \sum_{\alpha=1}^{N} \int_{r=a} \mathrm{d}S^\alpha \mathbf{n} \cdot [-\mu q(\mathbf{y}^\alpha + \mathbf{r})\mathbf{I} + \boldsymbol{\tau}(\mathbf{y}^\alpha + \mathbf{r})] = 0, \tag{21}$$

where $v = \pi a^2$ is the particle volume (area), the unit normal is directed into the continuous phase as before, and the contribution of the cell boundary vanishes by the assumed periodicity of q and $\mathbf{v}$. On the other hand, the hydrodynamic force $\mathbf{F}^\alpha$ on the αth particle is given by

$$\mathbf{F}^\alpha = \int_{r=a} \mathrm{d}S^\alpha \mathbf{n} \cdot [-p(\mathbf{y}^\alpha + \mathbf{r})\mathbf{I} + \boldsymbol{\tau}(\mathbf{y}^\alpha + \mathbf{r})] = -v\nabla P_\infty + \int_{r=a} \mathrm{d}S^\alpha \mathbf{n} \cdot (-\mu q\mathbf{I} + \boldsymbol{\tau}). \tag{22}$$

By using this relation the integrals in (21) can be eliminated to find

$$\nabla P_\infty = -\frac{1}{\mathscr{V}} \sum_{\alpha=1}^{N} \mathbf{F}^\alpha + \left(1 - \frac{Nv}{\mathscr{V}}\right)\mathbf{G} = -\frac{1}{\mathscr{V}} \sum_{\alpha=1}^{N} (\mathbf{F}^\alpha + v\mathbf{G}) + \mathbf{G}, \tag{23}$$

which demonstrates the hydrostatic nature of the field P_∞. Up to an inconsequential constant, thus we may take

$$P_\infty = \mathbf{x} \cdot \left(-\frac{1}{\mathscr{V}} \sum_{\alpha=1}^{N} \mathbf{f}^\alpha + \mathbf{G}\right), \tag{24}$$

where $\mathbf{f}^\alpha = \mathbf{F}^\alpha + v\mathbf{G}$ is the net fluid force on the particles, i.e., the hydrodynamic force plus buoyancy. It has been shown by Sangani and Yao (1988) that, with P_∞ as given by (24), q and $\mathbf{v}$ are indeed periodic. Upon substitution of this relation into the Stokes equation for an incompressible fluid we have

$$-\nabla q + \nabla^2 \mathbf{v} = -\frac{1}{\mu} \frac{1}{\mathscr{V}} \sum_{\alpha=1}^{N} \mathbf{f}^\alpha \equiv \mathbf{C}(N), \tag{25}$$

where the vector $\mathbf{C}$ depends on the particle configuration but, for each configuration, is a constant.

4. Average pressure

The average of the continuous-phase pressure is found by averaging (19):

$$\langle p \rangle = \langle P_\infty \rangle + \mu \langle q \rangle. \tag{26}$$

The second term is periodic and can be evaluated as in Section 2 while, for the first term, we use the definition (4) of ensemble average directly.

According to (12) and (13), the calculation of $\langle q \rangle$ requires the knowledge of q in the neighborhood of each particle. Thus, in the neighborhood of the generic particle α, we set

$$\tilde{q} = q + \mathbf{C} \cdot \mathbf{x} = q + \mathbf{C} \cdot (\mathbf{y}^\alpha + \mathbf{r}), \tag{27}$$

with which

$$-\nabla \tilde{q} + \nabla^2 \mathbf{v} = 0. \tag{28}$$

Due to the incompressibility of the continuous phase, $\tilde{q}$ is harmonic so that the last term in (12) vanishes and it is sufficient to calculate the surface integrals $L^\alpha_{\mathbf{k}}$ defined in (13); the same procedure can be followed for the second term $\mathbf{C} \cdot (\mathbf{y}^\alpha + \mathbf{r})$. We write

$$\tilde{q} = \frac{v}{a^2} \sum_{-\infty}^{\infty} q^\alpha_n, \tag{29}$$

where v is the kinematic viscosity of the continuous phase and

$$q^\alpha_n = s^n (P^\alpha_n \cos n\theta + \widetilde{P}^\alpha_n \sin n\theta), \tag{30}$$

a relation which also holds for $n = 0$. Here and in the following we set

$$s = \frac{r}{a}, \tag{31}$$

where r is the radial distance from the particle center, and it proves convenient to measure the angle $\theta = 0$ from the direction of the vector $\mathbf{k}$.

Some aspects of the calculation are dealt with in Appendix D and a considerably more detailed document is available from the author; here we only present the final result, which is

$$\beta_C \langle p \rangle = \mathbf{x} \cdot \left(-\frac{N}{\mathscr{V}} \bar{\mathbf{f}}_a + \mathbf{G} \right) - \frac{\mu v}{a^2} \mathscr{S}_1 (n v \bar{q}_0) + \frac{1}{2} v \mu v \nabla \cdot \sum_{l=1}^{\infty} \frac{1}{l!} \left(-\frac{a^2}{2} \right)^{l-1} (\nabla \cdot)^{(l-1)}$$
$$\times \left[\mathscr{S}_{l+1} \left(n \overline{\nabla^{(l)}_r (q_l + s^{2l} q_{-l})} \right) + 4l (a^2 \nabla^2)^{-1} \mathscr{S}_l \left(n \overline{\nabla^{(l)}_r (s^{2l} q_{-l})} \right) \right]. \tag{32}$$

In this equation n is the local particle number density given by

$$n(\mathbf{x}) = \frac{1}{N!} \int \mathrm{d}\mathscr{C}^N P(N) \sum_{\alpha=1}^{N} \delta(\mathbf{x} - \mathbf{y}^\alpha), \tag{33}$$

and overlines denote *particle averages* defined, for a quantity g^α pertaining to the particle α as a whole, by (see Zhang and Prosperetti, 1994, 1997; Prosperetti, 1998; Marchioro et al., 1999, 2000)

$$n(\mathbf{x}) \bar{g}(\mathbf{x}) = \frac{1}{N!} \int \mathrm{d}\mathscr{C}^N P(N) \left[\sum_{\alpha=1}^{N} \delta(\mathbf{x} - \mathbf{y}^\alpha) g^\alpha(N) \right]; \tag{34}$$

$\bar{\mathbf{f}}_a$ is defined by

$$\bar{\mathbf{f}}_a = \frac{1}{N} \int \mathrm{d}^2 x n(\mathbf{x}) \bar{\mathbf{f}}(\mathbf{x}), \tag{35}$$

and represents therefore the average force over all the particles in the cell. The operator $(\nabla^2)^{-1}$ is the formal inverse of ∇^2 in the space of functions periodic on the fundamental cell. Finally, the $\mathscr{S}_l$ are differential operators defined by

$$\mathscr{S}_l(a^2\nabla^2) = \sum_{k=0}^{\infty} \frac{1}{k!(l+k)!}\left(\frac{a^2\nabla^2}{4}\right)^k; \tag{36}$$

for an interpretation of these operators and some of their properties see Appendix C where, in particular, it is shown that the particle volume fraction β_{D} is given by

$$\beta_{\mathrm{D}} = \mathscr{S}_1(nv) \simeq \left(1 + \frac{a^2}{8}\nabla^2 + \cdots\right)(nv). \tag{37}$$

For a spatially uniform system all derivatives vanish, $\mathscr{S}_1$ reduces to the identity, and one finds $\beta_{\mathrm{D}} = nv$.

Note that, in (32), the differential operators $\mathbf{\nabla}_r$ only occur under the particle average overline and operate on homogeneous polynomials of degree l in the local variable $\mathbf{r}/a$ for each particle; hence, before averaging, both $\mathbf{\nabla}_r^{(l)}(q_l^{\alpha} + s^{2l}q_{-l})$ and $\mathbf{\nabla}_r^{(l)}(s^{2l}p_{-l})$ are in fact constant tensors of degree l. It can be shown that they are related to the moments of the hydrodynamic traction acting on the surface of the particles (see Prosperetti, 2003). The other operator $(\mathbf{\nabla}\cdot)^{(l-1)}$ carrying no subscript is the divergence operator repeated $l-1$ times and acting on the field variable $\mathbf{x}$, the dependence on which arises from the particle average operation (34).

In single-phase incompressible fluid mechanics, the momentum equation is invariant if the pressure p is replaced by $p + \psi(\mathbf{x})$, where ψ is an arbitrary harmonic function, and the body force $\mathbf{G}$ is replaced by $\mathbf{G} + \mathbf{\nabla}\psi$. This *gauge invariance* property is physically significant as it embodies the notion that, in an incompressible medium, the pressure is not a thermodynamic variable but has only the role to enforce incompressibility. In Marchioro et al. (1999) it was argued that, in a two-phase disperse system in which the two phases are individually incompressible, the quantity to be identified as the *mixture pressure*, p_{m}, should enjoy the same property; the reference cited contains an extensive discussion justifying the procedure. Here we simply note that, in the present two-dimensional system, the definition of p_{m} given in Marchioro et al. (1999) should be modified to

$$p_{\mathrm{m}} = \beta_{\mathrm{C}}\langle p\rangle + \frac{\mu v}{a^2}\left(1 + \frac{a^2}{8}\nabla^2\right)(nv\bar{q}_0) + \frac{1}{4}a^2\mathbf{\nabla}\cdot\left(\overline{n\int_{r=a} \mathrm{d}S(-\mathbf{n})p}\right) + \frac{1}{2}a^3\mathbf{\nabla\nabla}:\left(\overline{n\int_{r=a}\mathrm{d}S\left(\mathbf{nn} - \frac{1}{2}\mathbf{I}\right)p}\right) + \cdots \tag{38}$$

It is readily verified that, with this definition, up to higher-order terms, p_{m} is replaced by $p_{\mathrm{m}} + \psi$ if the continuous-phase pressure inside the integrals is replaced by $p + \psi$. [3] A simple calculation shows that the first three terms of the form $\mathscr{S}_{l+1}(\overline{n\mathbf{\nabla}_r^{(l)}(q_l + s^{2l}q_{-l})})$ in the right-hand side of (32) coincide with those shown in (38). Thus, similarly to Tanksley and Prosperetti (2001), we are led to *identify* the mixture pressure with the quantity

[3] Of course, the only regular periodic harmonic function is a constant, but the analysis of Marchioro et al. (1999) leading to (38) is general and does not presuppose a periodic system.

$$p_{\rm m} = \beta_{\rm C}\langle p\rangle + \frac{\mu v}{a^2}S_1(nv\bar{q}_0) - \frac{1}{2}v\mu v\sum_{l=1}^{\infty}\frac{1}{l!}\left(-\frac{a^2}{2}\right)^{l-1}S_{l+1}(\nabla\cdot)^l\left[n\left(\overline{\nabla_r^{(l)}(q_l^\alpha + s^{2l}q_{-l})}\right)\right]. \tag{39}$$

With this *definition*, (32) gives the following *result* for the mixture pressure:

$$p_{\rm m} = \mathbf{x}\cdot\left(-\frac{N}{\mathscr{V}}\bar{\mathbf{f}}_{\rm c} + \mathbf{G}\right) + 2\pi\mu v(\nabla^2)^{-1}\sum_{l=1}^{\infty}\frac{1}{(l-1)!}\left(-\frac{a^2}{2}\right)^{l-1}S_l(\nabla\cdot)^l\left(n\overline{\nabla_r^{(l)}(s^{2l}q_{-l})}\right). \tag{40}$$

It should be noted that this result for a *mixture-level quantity* only contains decaying harmonics, which represent the effect of the particles on the flow. In the next section it will be found that the average volumetric flux—which is unquestionably a mixture quantity—also possesses this feature. At a formal level, this circumstance strengthens the identification of (39) with the mixture pressure.

5. Average velocities

We now turn to a calculation of the average velocities, starting with $\mathbf{u}_{\rm D}(\mathbf{x})$, the velocity field of the particle material (which, in general, will be different from the velocity of the particle center). Since the particles are assumed rigid, when the point $\mathbf{x}$ is inside particle α we may write

$$\mathbf{u}_{\rm D}(\mathbf{x}) = \mathbf{w}^\alpha + \boldsymbol{\Omega}^\alpha\times(\mathbf{x}-\mathbf{y}^\alpha), \quad |\mathbf{x}-\mathbf{y}^\alpha|\leqslant a, \tag{41}$$

where $\mathbf{w}^\alpha$ and $\boldsymbol{\Omega}^\alpha$ are the translational and angular velocity of the αth particle with respect to the assumed reference frame. The phase-ensemble average of the disperse-phase velocity $\mathbf{u}_{\rm D}$ is defined by a relation similar to (4), namely

$$\beta_{\rm D}\langle\mathbf{u}_{\rm D}\rangle(\mathbf{x}) = \frac{1}{N!}\int {\rm d}\mathscr{C}^N P(N)\chi(\mathbf{x};N)\mathbf{u}_{\rm D}(\mathbf{x};N). \tag{42}$$

Note that, from (17), the no-slip condition at the particle surface requires that

$$\mathbf{v}(\mathbf{y}^\alpha+\mathbf{r}) = \mathbf{w}_0^\alpha - \gamma\cdot\mathbf{r} + \boldsymbol{\Omega}^\alpha\times\mathbf{r}, \tag{43}$$

where

$$\mathbf{w}_0^\alpha = \mathbf{w}^\alpha - \mathbf{U}_\infty(\mathbf{y}^\alpha). \tag{44}$$

Thus, for $\mathbf{v}$ to be periodic, $\mathbf{w}_0^\alpha$ must be the same for all the images of each α particle in all the cells (of course, the two $\mathbf{w}_0$ corresponding to different particles in the *same* cell are unrelated). We thus conclude that $\mathbf{u}_{\rm D}$ cannot be periodic, although $\mathbf{u}_{\rm D} - \mathbf{U}_\infty(\mathbf{x})$ is, and can therefore be expanded in a Fourier series:

$$\beta_{\rm D}(\mathbf{x})(\langle\mathbf{u}_{\rm D}\rangle - \mathbf{U}_\infty) = \mathbf{u}_0 + \sum_{\mathbf{k}\neq 0}\mathbf{u}_{\mathbf{k}}\exp(-{\rm i}\mathbf{k}\cdot\mathbf{x}), \tag{45}$$

where

$$\mathbf{u}_0 = \frac{1}{\mathscr{V}}\int {\rm d}^2x\beta_{\rm D}(\mathbf{x})(\langle\mathbf{u}_{\rm D}\rangle - \mathbf{U}_\infty), \quad \mathbf{u}_{\mathbf{k}} = \frac{1}{\mathscr{V}}\int {\rm d}^2x\exp({\rm i}\mathbf{k}\cdot\mathbf{x})\beta_{\rm D}(\mathbf{x})(\langle\mathbf{u}_{\rm D}\rangle - \mathbf{U}_\infty), \tag{46}$$

or, from (42),

$$\mathbf{u}_0 = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(N)\left[\frac{1}{\mathscr{V}}\int \mathrm{d}^2x\chi(\mathbf{u}_\mathrm{D} - \mathbf{U}_\infty)\right], \tag{47}$$

$$\mathbf{u}_\mathbf{k} = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(N)\left[\frac{1}{\mathscr{V}}\int \mathrm{d}^2x\chi\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})(\mathbf{u}_\mathrm{D} - \mathbf{U}_\infty)\right]. \tag{48}$$

It is shown in Appendix D that the final result is

$$\beta_\mathrm{D}\langle\mathbf{u}_\mathrm{D}\rangle = \beta_\mathrm{D}\mathbf{U}_\infty + \mathscr{S}_1(a^2\nabla^2)(nv\overline{\mathbf{w}}) + a^2\mathscr{S}_2(a^2\nabla^2)\left[\nabla\times(nv\overline{\boldsymbol{\Omega}}) - \boldsymbol{\gamma}\cdot\nabla(vn)\right]. \tag{49}$$

For a spatially homogeneous system this relation simply gives $\langle\mathbf{u}_\mathrm{D}\rangle = \overline{\mathbf{w}}$. The result (49) shows however that, in general, the ensemble average velocity of the *particle material* differs from the average *translational velocity* of the particle centers $\overline{\mathbf{w}}$ (see also Lhuillier, 1992; Zhang and Prosperetti, 1994).

The mixture velocity, or total volumetric flux, is defined by

$$\mathbf{u}_\mathrm{m} = \beta_\mathrm{C}\langle\mathbf{u}_\mathrm{C}\rangle + \beta_\mathrm{D}\langle\mathbf{u}_\mathrm{D}\rangle. \tag{50}$$

The contribution of the disperse phase was evaluated in (49). For the continuous phase, we use the Stokes equation (25) to express $\nabla^2\mathbf{v}$ in the last term of (12) to find

$$\begin{aligned}\mathbf{V}_\mathbf{k}(N) &\equiv \int_\mathscr{L} \mathrm{d}^2x\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\mathbf{v}(\mathbf{x};N)\\ &= \frac{1}{k^2}\sum_{\alpha=1}^{N}\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{y}^\alpha)\left\{\int_{r=a}\mathrm{d}S^\alpha[\mathrm{i}(\mathbf{n}\cdot\mathbf{k})\mathbf{v} - (\mathbf{n}\cdot\nabla)\mathbf{v}]\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\right\} - \frac{1}{k^2}\int_\mathscr{L}\mathrm{d}^2x\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})\nabla q.\end{aligned} \tag{51}$$

Since q is harmonic, ∇q also is and the last integral can be treated as done before in Section 4. The first term in the first integral can readily be evaluated by using the boundary condition (43) while, for the second one, we use the representation of the velocity field given in Appendix A.

Some details of the calculation are given in Appendix D; here we simply quote the final result, which is

$$\begin{aligned}\mathbf{u}_\mathrm{m} = {} & \mathbf{u}_\infty - 2\pi va^2(\nabla^2)^{-1}(\mathbf{I}\nabla^2 - \nabla\nabla)\cdot\sum_{l=1}^{\infty}\frac{1}{(l-1)!}\left(-\frac{a^2}{2}\right)^{l-1}(\nabla_x\cdot)^{l-1}\\ & \times\left\{(l+1)\mathscr{S}_{l+1}\left(n\overline{\nabla_r^{(l)}(s^{2l}\phi_{-l})}\right) + \left((a^2\nabla^2)^{-1}\mathscr{S}_l - \frac{1}{4}\mathscr{S}_{l+1}\right)n\overline{\nabla_r^{(l)}(s^{2l}q_{-l})}\right\} + 2\pi v\left(n\overline{\nabla_r(s^2\phi_{-1})}\right)_0\\ & + 2\pi\frac{v}{a^2}(\nabla^2)^{-1}\left(n\overline{\nabla_r(s^2q_{-1})}\right)_0 - \frac{1}{2\mu}(\nabla^2)^{-1}\nabla\times\mathscr{S}_1(n\overline{\mathbf{L}}),\end{aligned} \tag{52}$$

where $\mathbf{L}^\alpha$ is the hydrodynamic couple acting on the αth particle.

It may be noted that $\mathbf{u}_\mathrm{m}$ is a *mixture quantity*, just as the mixture pressure previously defined in (40): it is noteworthy that the results for both quantities share the property of containing only decaying harmonics.

6. Mixture momentum balance

For a pure fluid, in the absence of inertia, the combination $-\nabla p + \mu\nabla^2\mathbf{u}$ equals the negative of the body force. In a mixture one would expect to find, in addition to the latter, the divergence of a term related to the fluid-particle interaction, to which we may refer as the *particle stress*. This quantity is expected to appear in the form of a divergence because, being an internal force of the system, its integration over a finite volume must reduce to a boundary contribution. We are thus led to consider the combination $-\nabla p_m + \mu\nabla^2\mathbf{u}_m$, with p_m given by (40) and $\mathbf{u}_m$ by (52). A simple calculation shows that

$$-\nabla p_m + \mu\nabla^2\mathbf{u}_m = -\mathbf{G} - 2\pi\mu v S_1\left(n\overline{\nabla_r(s^2 q_{-1})}\right) - \nabla\cdot\mathsf{S} - \nabla\times\left[\frac{1}{2}\mathscr{S}_1(n\overline{\mathbf{L}}) + \nabla\times(\mathbf{U}^\phi + \mathbf{U}^p)\right], \tag{53}$$

where

$$\mathsf{S} = 2\pi\mu v\sum_{l=2}^{\infty}\frac{1}{(l-1)!}\left(-\frac{a^2}{2}\right)^{l-1}\mathscr{S}_l(\nabla\cdot)^{(l-2)}\left(n\overline{\nabla_r^{(l)}(s^{2l}q_{-l})}\right), \tag{54}$$

$$\mathbf{U}^\phi = -2\mu v v\sum_{l=1}^{\infty}\frac{l+1}{(l-1)!}\left(-\frac{a^2}{2}\right)^{l-1}\mathscr{S}_{l+1}(\nabla\cdot)^{(l-1)}\left(n\overline{\nabla_r^{(l)}(s^{2l}\phi_{-l})}\right), \tag{55}$$

$$\mathbf{U}^p = \frac{1}{2}\mu v v\sum_{l=1}^{\infty}\frac{1}{(l-1)!}\left(-\frac{a^2}{2}\right)^{l-1}\mathscr{S}_{l+1}(\nabla\cdot)^{(l-1)}\left(n\overline{\nabla_r^{(l)}(s^{2l}q_{-l})}\right). \tag{56}$$

It may be recalled that, as observed earlier after Eq. (37), the differentiated terms with an overbar appearing in these definitions are actually constants related to the average force multipoles on the particles.

We now examine the quantities (54)–(56) in turn.

From the expression for the velocity field given in Appendix A, it is easy to show that the hydrodynamic force equals

$$\mathbf{F}^\alpha = -v\mathbf{G} - 2\pi\mu v\nabla_r(s^2 q_{-1}^\alpha), \tag{57}$$

and, therefore,

$$\mathbf{G} + 2\pi\mu v\mathscr{S}_1\left(n\overline{\nabla_r(s^2 q_{-1})}\right) = \beta_C\mathbf{G} - \mathscr{S}_1(n\overline{\mathbf{F}}) = \beta_C\mathbf{G} - \frac{1}{v}\int_{|\mathbf{r}|\leqslant a}\mathrm{d}^2 r n(\mathbf{x}+\mathbf{r})\overline{\mathbf{F}}(\mathbf{x}+\mathbf{r}), \tag{58}$$

where we have used (37) and the representation (C.4) of the operator $\mathscr{S}_1$. In particular, if the particle inertia is unimportant, the hydrodynamic force must balance the external applied force $\mathbf{F}_e$ and, when the latter is the same for all the particles,

$$\mathbf{G} + 2\pi\mu v\mathscr{S}_1\left(n\overline{\nabla_r(s^2 q_{-1})}\right) = \beta_C\mathbf{G} + \frac{\mathbf{F}_e}{v}\beta_D. \tag{59}$$

Thus, the combination of these two terms gives the correct force per unit volume acting locally on the mixture of the two phases.

The definition (54) of the two-tensor S shows that the two free tensorial indices are $\nabla_r\nabla_r$ under the particle-average sign: it follows that this tensor is symmetric and traceless, since $s^{2l}q_{-l}$ is harmonic. In a spatially uniform system, the summation (54) reduces to the first term $\mathscr{S}_2$ since all others are differentiated at least once with respect to the field variable $\mathbf{x}$; as Eq. (C.5) shows, in this case, $\mathscr{S}_2$ simply equals $\frac{1}{2}$. Thus we conclude that, for a spatially uniform system,

$$\mathsf{S} = -\frac{\pi}{2} a^2 \mu v n \overline{\nabla\nabla(s^4 q_{-2})}. \tag{60}$$

But the quantity in the right-hand side is readily shown to be just the average stresslet acting on the particles which, in two dimensions, is defined by

$$\overline{\frac{1}{2}\int_{r=a} \mathrm{d}S\left[x_i\sigma_{jk} + x_j\sigma_{ik} - \delta_{ij}x_\ell\sigma_{k\ell}\right]n_k} = -\frac{\pi}{2}a^2\mu v \overline{\partial_i\partial_j(s^4 q_{-2})}; \tag{61}$$

we thus recover the well-known result given in Batchelor (1970).

The last group of terms in the momentum balance (53) constitutes the antisymmetric part of the particle stress. The first term, $\mathbf{L}$, is the mean hydrodynamic couple per unit volume acting on the particles:

$$\frac{1}{2}\mathscr{S}_1(n\overline{\mathbf{L}}) = \frac{1}{2v}\int_{|\mathbf{r}|\leqslant a} \mathrm{d}^2 r n(\mathbf{x}+\mathbf{r})\overline{\mathbf{L}}(\mathbf{x}+\mathbf{r}), \tag{62}$$

where, again, the representation (C.4) of the operator $\mathscr{S}_1$ has been used. If the integral is approximated by $\frac{1}{2}vn(\mathbf{x})\overline{\mathbf{L}}(\mathbf{x})$, which is permissible in a uniform suspension, we recover another result in Batchelor (1970).

In the absence of inertia, whenever the particles are couple free, the antisymmetric contribution (62) to the particle stress vanishes. However, unlike the spatially uniform case, this fact does not automatically ensure symmetry of the particle stress tensor as the last two terms, $\mathbf{U}^\phi$ and $\mathbf{U}^p$, can also induce an antisymmetric contribution in the presence of spatial non-uniformities. If their expressions (55) and (56) are truncated to the first term, and $\mathscr{S}_2$ is approximated by its leading term $\frac{1}{2}$, we have

$$\mathbf{U}^p + \mathbf{U}^\phi = \frac{1}{4}\mu v v\left[n\overline{\nabla_r(s^2 p_{-1} - 8\phi_{-1})}\right] = \frac{1}{4}na^2\overline{\int_{r=a}\mathrm{d}S(\mathbf{I}-\mathbf{nn})\cdot\boldsymbol{\tau}}. \tag{63}$$

To leading order, therefore, this term is proportional to the average contribution of the tangential traction to the hydrodynamic force on the particle. At low Reynolds numbers, the tangential and normal tractions contribute similarly to this force and, therefore, the integral in (63) may be estimated to be of the order of the hydrodynamic force. As this force may be expected to be proportional to the relative velocity multiplied by a function of the volume fraction, we conclude that this term will influence the mixture momentum in the presence of non-uniformities in either one of these quantities. Since a force (per unit length) acting on the particles induces a relative interphase velocity $\Delta u \sim F/\mu$, (an estimate which is confirmed by the results of the next section), in order of magnitude, we have $\nabla\times\nabla\times(\mathbf{U}^p+\mathbf{U}^\phi) \sim \beta_\mathrm{D}\mu\Delta u/L^2$, which is comparable to $(\mu_\mathrm{eff}-\mu)\nabla^2 u_\mathrm{m}$, where μ_eff is the effective viscosity of the suspension.

Eq. (53) is the momentum balance for the suspension. Upon equating the particle average of (57) to the average external force applied to the particles, $\overline{\mathbf{F}}_{\mathrm{e}}$, one finds the momentum equation for the particle phase:

$$\overline{\mathbf{F}}_{\mathrm{e}} = -v\mathbf{G} - 2\pi\mu v \overline{\nabla_r (s^2 q^{\alpha}_{-1})}. \tag{64}$$

A similar procedure applied to the average couple acting on the particles gives the angular momentum equation. As shown in several papers (e.g. Zhang and Prosperetti, 1994, 1997; Marchioro et al., 1999, 2000), averaging of the total mass and particle number conservation relations gives

$$\nabla \cdot \mathbf{u}_{\mathrm{m}} = 0, \quad \frac{\partial n}{\partial t} + \nabla \cdot (n\overline{\mathbf{w}}) = 0. \tag{65}$$

Provided closure relations for the right-hand sides of (54)–(56), and (64) can be formulated, Eqs. (53), (64), and (65) constitute a complete "two-fluid" model of the suspension. As shown in Marchioro et al. (2001) and Wang and Prosperetti (2001), direct numerical simulation is a powerful tool for the systematic development of such closure relations; the results presented in this paper are a necessary basis for this task.

7. A cell model

The main purposes of this work were, first, to elucidate the nature of the stress in a suspension of disks, and, second, to prepare the ground for a numerical treatment similar to that provided for a suspension of spheres in our earlier papers (Marchioro et al., 2000, 2001; Wang and Prosperetti, 2001). However, in conclusion, it may be of some interest to use the previous analysis to implement a simple cell model. The limitations of such models are well known, and the reason to develop one here is only to estimate some of the coefficients that would arise in a final closed description of a two-dimensional suspension. It may be noted that an effective medium approximation might be more accurate (see e.g. Spelt et al., 2001). However, its application requires that the form of the averaged equations be known and, furthermore, that it be possible to solve them in the clear-fluid zone surrounding the "test" particle. While these two conditions are easily met for a spatially uniform system, they clearly are not in the present case.

The results of the previous section are expressed as series the terms of which contain spatial derivatives of increasingly higher order. In order to preserve the mathematical nature of the standard equations of fluid mechanics, we retain only terms contributing at most second-order derivatives of the velocity to the final result and also, for simplicity, neglect the body force $\mathbf{G}$ on the fluid and consider couple-free and inertialess particles.

To construct the model, we consider a particle immersed in Stokes flow in a cylindrical region of clear fluid of radius R chosen so that $(\pi a^2/\pi R^2) = \beta_{\mathrm{D}}$, the volume fraction of the suspension. The particle is in motion with velocity $\mathbf{w}$ and rotates with angular velocity $\mathbf{\Omega}$ and we require that, at the surface of the region of radius R, the velocity equal

$$\mathbf{U} = \mathbf{u}_{\mathrm{m}} + (\mathbf{r} \cdot \nabla)\mathbf{u}_{\mathrm{m}} + \frac{1}{2}(\mathbf{rr} : \nabla\nabla)\mathbf{u}_{\mathrm{m}}, \tag{66}$$

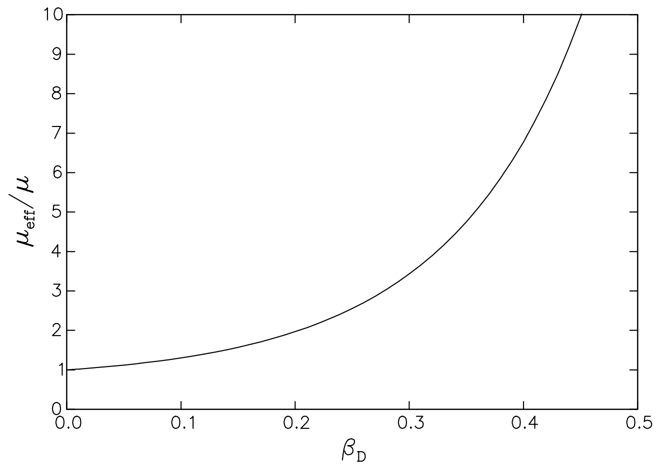

Fig. 2. Normalized effective viscosity according to the cell model of Section 7.

where $\mathbf{u}_m$ and its gradients are evaluated at the position occupied by the center of the particle.

From the solution of the Stokes equation satisfying (66) one can calculate the various quantities introduced in the previous section. The symmetric part of the stress is found to be

$$\mathsf{S} = [\mu_{eff}(\beta_D) - \mu](\nabla\mathbf{u}_m + \nabla\mathbf{u}_m^T), \tag{67}$$

where the effective viscosity is given by

$$\frac{\mu_{eff}}{\mu} = 1 + 2\frac{1 + \beta_D + \beta_D^2}{(1 - \beta_D)^3} nv. \tag{68}$$

In particular, for $\beta_D \to 0$, one recovers the Einstein viscosity correction for the two-dimensional case, $\mu_{eff}/\mu = 1 + 2nv$, which is a known result (see e.g. Belzons et al., 1981; Brady, 1984). [4] A graph of μ_{eff}/μ as given by (68) is shown in Fig. 2. Dodd et al. (1995) have calculated numerically in two different ways the effective viscosity of a suspension of discs and a comparison with some of their results with the present ones is shown in Table 1. It is seen that the agreement is reasonable up to about 10–15%, after which (68) increasingly overestimates the effective viscosity.

The simple implementation of the cell model used here does not give rise to additional terms in the right-hand side of (67) that were found necessary to reproduce the direct numerical simulations of Marchioro et al. (2001). Such terms would be proportional to the symmetric parts of $\nabla(\overline{\mathbf{w}} - \mathbf{u}_m)$ and $(\overline{\mathbf{w}} - \mathbf{u}_m)\nabla\beta_D$ and would thus introduce other "viscosities" expressing the resistance of the mixture to the rate of deformation of its microstructure.

From the requirement (64) that the external force balance the hydrodynamic force, we find

$$\overline{\mathbf{F}}_e = -\frac{4\pi\mu}{M(\beta_D)}(\mathbf{u}_m - \overline{\mathbf{w}}) - \frac{\mu v}{(1 + \beta_D)M(\beta_D)}\nabla^2\mathbf{u}_m, \tag{69}$$

[4] The dilute limit result can be derived directly with no recourse to the cell model by identifying the velocity field incident on the particle with $\mathbf{u}_m$.

Table 1
Comparison between the effective viscosity given by (68) and the results found by two different methods of Dodd et al. (1995)

β_D	μ_{eff}/μ		
	Present	Dodd et al. I	Dodd et al. II
0.05	1.123	1.13	–
0.10	1.305	1.28	–
0.15	1.573	1.40	1.4
0.20	1.969	1.67	–
0.30	3.431	2.27	2.2

where the mobility M is given by

$$M = -\frac{1}{2}\log\beta_D - \frac{1-\beta_D}{1+\beta_D}. \tag{70}$$

The second term in (69) is analogous to the Faxén contribution to the force on a sphere placed in a non-uniform Stokes flow. Howells (1974, 1998) derived an asymptotic expression for the mobility of a spatially uniform random arrangement of discs:

$$M - \frac{1}{2}\log M + \gamma - \frac{0.470}{M} + O(M^{-2}) = -\frac{1}{2}\log\beta_D, \tag{71}$$

where γ is Euler's constant. The two results agree to leading order, $M \simeq -\frac{1}{2}\log\beta_D$. A comparison beyond this limit can be carried out numerically, but the series (71) can only be used for large M, which implies a very small volume fraction given that $\beta_D \sim \exp(-2M)$. For example, for $\beta_D = 3.21\times 10^{-3}$, from (71), M has the relatively small value of 3.00, while (70) gives $M = 1.88$. It can be concluded that the two results diverge rapidly from each other as soon as terms beyond the first one become important. As $\beta_D \to 0$, M diverges, which is a manifestation of the so-called Stokes paradox of two-dimensional Stokes flow. For force-free particles, (69) gives

$$\overline{\mathbf{w}} = \mathbf{u}_m + \frac{a^2}{4(1+\beta_D)}\nabla^2\mathbf{u}_m, \tag{72}$$

which shows that, as mentioned in Section 1, the average particle velocity equals the mixture velocity in the case of uniform shear.

The balance of the hydrodynamic and external applied couples requires that

$$\overline{\mathbf{L}}_e = 4\mu v\frac{\overline{\boldsymbol{\Omega}} - \frac{1}{2}\nabla\times\mathbf{u}_m}{M_R}, \tag{73}$$

where the rotational mobility M_R is found to be given by

$$M_R = 1 - \beta_D. \tag{74}$$

This simple result is found to provide a very good fit to the numerical calculations of Dodd et al. (1995), from which it differs by less than 10% over the range $0\% \leqslant \beta_D \leqslant 60\%$. Again, (69) and (73) do not include several other terms, such as $(\nabla\mathbf{u}_m + \nabla\mathbf{u}_m^T)\cdot\nabla\beta_D$, $[(\overline{\mathbf{w}} - \mathbf{u}_m)\cdot\nabla]\nabla\beta_D$, etc. that were found in Marchioro et al. (2001).

The components of the antisymmetric stress are

$$\mathbf{U}^{\phi} = \frac{\mu v n}{(1+\beta_{\mathrm{D}})M}(\mathbf{u}_{\mathrm{m}} - \overline{\mathbf{w}}), \quad \mathbf{U}^{p} = -\frac{1}{2}\frac{\mu v n}{M}(\mathbf{u}_{\mathrm{m}} - \overline{\mathbf{w}}), \tag{75}$$

both of which also vanish in the dilute limit. Additional terms that might arise here, similar to those mentioned in connection with (69), would give rise to higher derivatives in the momentum equation and, therefore, would not contribute to the final result in the present approximation. Furthermore, by using (69) and truncating the result so as to retain only second-order derivatives, we have

$$\begin{aligned}\frac{1}{v}\int_{r\leqslant a} \mathrm{d}^2 x n(\mathbf{x}+\mathbf{r})\overline{\mathbf{F}}(\mathbf{x}+\mathbf{r}) &\simeq \left(1+\frac{a^2}{8}\nabla^2\right)[n(\mathbf{x})\overline{\mathbf{F}}(\mathbf{x})] \\ &\simeq \frac{4\pi\mu n}{M}(\mathbf{u}_{\mathrm{m}} - \overline{\mathbf{w}}) + \mu\frac{\beta_{\mathrm{D}}}{(1+\beta_{\mathrm{D}})M}\nabla^2\mathbf{u}_{\mathrm{m}} + \nabla^2\left[\frac{\beta_{\mathrm{D}}}{2M}(\mathbf{u}_{\mathrm{m}} - \overline{\mathbf{w}})\right]. \end{aligned}\tag{76}$$

Upon combining these contributions, and again neglecting terms with derivatives beyond the second order and external couples, Eq. (53) becomes

$$\begin{aligned}&-\nabla p_{\mathrm{m}} + \nabla\cdot[\mu_{\mathrm{eff}}(\nabla\mathbf{u}_{\mathrm{m}} + \nabla\mathbf{u}_{\mathrm{m}}^{\mathrm{T}})] + \mu\nabla\times\nabla\times\left[\frac{\beta_{\mathrm{D}}}{2M}\frac{1-\beta_{\mathrm{D}}}{1+\beta_{\mathrm{D}}}(\mathbf{u}_{\mathrm{m}} - \overline{\mathbf{w}})\right] - \frac{4\pi\mu n}{M}(\mathbf{u}_{\mathrm{m}} - \overline{\mathbf{w}}) \\ &\quad - \mu\frac{\beta_{\mathrm{D}}}{(1+\beta_{\mathrm{D}})M}\nabla^2\mathbf{u}_{\mathrm{m}} - \mu\nabla^2\left[\frac{\beta_{\mathrm{D}}}{2M}(\mathbf{u}_{\mathrm{m}} - \overline{\mathbf{w}})\right] = 0. \end{aligned}\tag{77}$$

In the special case in which the force acting on each particle is the same, $\mathbf{F}_{\mathrm{e}}$, as for example in the case of sedimentation, the right-hand side of (76) reduces to $-(\beta_{\mathrm{D}}/v)\mathbf{F}_{\mathrm{e}}$ and, again omitting higher-order derivatives, this equation simplifies to the form

$$-\nabla p_{\mathrm{m}} + \nabla\cdot[\mu_{\mathrm{eff}}(\nabla\mathbf{u}_{\mathrm{m}} + \nabla\mathbf{u}_{\mathrm{m}}^{\mathrm{T}})] - \frac{1}{8\pi}\nabla\times\nabla\times\left[\beta_{\mathrm{D}}\frac{1-\beta_{\mathrm{D}}}{1+\beta_{\mathrm{D}}}\mathbf{F}_{\mathrm{e}}\right] + \frac{\beta_{\mathrm{D}}}{v}\mathbf{F}_{\mathrm{e}} = 0. \tag{78}$$

Upon taking the vector product of this relation with $\mathbf{F}_{\mathrm{e}}$, one finds

$$\mathbf{F}_{\mathrm{e}}\times\left[-\nabla p_{\mathrm{m}} + \nabla\cdot[\mu_{\mathrm{eff}}(\nabla\mathbf{u}_{\mathrm{m}} + \nabla\mathbf{u}_{\mathrm{m}}^{\mathrm{T}})] - \frac{1}{8\pi}(\mathbf{F}_{\mathrm{e}}\cdot\nabla)\nabla\left(\beta_{\mathrm{D}}\frac{1-\beta_{\mathrm{D}}}{1+\beta_{\mathrm{D}}}\right)\right] = 0. \tag{79}$$

It is evident here that, if $(\mathbf{F}_{\mathrm{e}}\cdot\nabla)\nabla\beta_{\mathrm{D}}$ has a non-zero component perpendicular to $\mathbf{F}_{\mathrm{e}}$, at least one of the first two terms in the square brackets must also have a non-zero component in the same direction, i.e., a motion in the plane normal to $\mathbf{F}_{\mathrm{e}}$ must take place. In other words, in a suspension with a horizontal and vertical inhomogeneity in the particle distribution, in addition to the vertical settling of the particles, a motion in the horizontal direction must also take place. This result, which is a consequence of the lack of symmetry of the particle stress, suggests that a state of uniform settling would be unstable to two-dimensional perturbations of the particle volume fraction. Note that, if β_{D} were only a function of the coordinate perpendicular to $\mathbf{F}_{\mathrm{e}}$, a motion transversal to $\mathbf{F}_{\mathrm{e}}$ would be induced simply by the term $\frac{\beta_{\mathrm{D}}}{v}\mathbf{F}_{\mathrm{e}}$ in (78). This situation would be similar to the well-known fact that a fluid the density of which varies in the horizontal direction cannot be at rest in a gravitational field. The effect induced by the antisymmetric stress in an additional one, which enters into play when β_{D} depends on both the horizontal and vertical coordinates.

Another interesting situation is a porous medium, for which $\overline{\mathbf{w}} = 0$. In this case, (77) becomes

$$-\nabla p_{\mathrm{m}} + \nabla \cdot [\mu_{\mathrm{eff}}(\nabla \mathbf{u}_{\mathrm{m}} + \nabla \mathbf{u}_{\mathrm{m}}^{\mathrm{T}})] + \mu \nabla \times \nabla \times \left[\frac{\beta_{\mathrm{D}}}{2M}\frac{1-\beta_{\mathrm{D}}}{1+\beta_{\mathrm{D}}}\mathbf{u}_{\mathrm{m}}\right] - \frac{4\pi\mu n}{M}\mathbf{u}_{\mathrm{m}}$$
$$- \mu \frac{\beta_{\mathrm{D}}}{(1+\beta_{\mathrm{D}})M}\nabla^2 \mathbf{u}_{\mathrm{m}} - \frac{1}{2}\mu\nabla^2\left(\frac{\beta_{\mathrm{D}}}{M}\mathbf{u}_{\mathrm{m}}\right) = 0. \tag{80}$$

When the particle volume fraction is uniform, this equation simplifies to

$$-\nabla p_{\mathrm{m}} + \mu\left(\frac{\mu_{\mathrm{eff}}}{\mu} - 2\frac{\beta_{\mathrm{D}}}{(1+\beta_{\mathrm{D}})M}\right)\nabla^2 \mathbf{u}_{\mathrm{m}} - \frac{4\pi\mu n}{M}\mathbf{u}_{\mathrm{m}} = 0, \tag{81}$$

which, for a state of uniform flow, reduces to a relation of the Darcy form. When $\nabla^2 \mathbf{u}_{\mathrm{m}} \neq 0$ we recover a Brinkman equation but, interestingly, the "Brinkman viscosity" that appears is neither the effective viscosity of the mixture μ_{eff}, nor that of the pure fluid. As a matter of fact, as shown in Wang and Prosperetti (2001), when all the proper terms are retained (i.e., not only those shown in (81), but also those that cannot be calculated from the cell model), there are additional contributions to the coefficient of $\nabla^2 \mathbf{u}_{\mathrm{m}}$. It is well known (see e.g. Martys et al., 1994) that a persistent difficulty in the use of the Brinkman equation is precisely related to uncertainties in the proper effective viscosity to use in it, which is believed to differ from both μ_{eff} and μ. Our result may give an explanation of this fact. Furthermore, the previous developments show that the Brinkman equation is a consequence of assuming a spatially uniform particle distribution, which will obviously be incorrect at a porous medium–clear fluid interface. This is precisely the situation which motivated the original introduction of Brinkman's equation, and the situation in which it is most frequently applied. A comparable situation was found in the three-dimensional case (Wang and Prosperetti, 2001). An investigation of the matter by means of methods more powerful than a cell model and, in particular, by numerical simulation, would be valuable in shedding light on the proper status of the Brinkman equation for a disordered medium. The results reported here form a necessary prerequisite for this work.

8. Conclusions

We have calculated the ensemble-average velocity and pressure in a periodic suspension of disks and, from these results, evaluated the particle stress. Unlike many other studies, we have accounted for the possibility of a spatial non-uniformity in the system finding results in close analogy with the corresponding ones for a suspension of spheres given in Tanksley and Prosperetti (2001). As in that case, it is found that the stress acquires an antisymmetric component induced by spatial non-uniformities even when no external couples act on the particles. General considerations (at the end of Section 6) and a cell model (in Section 7) suggest that this antisymmetric component will induce, for example, a horizontal flow in the settling of a suspension with gradients of the particle volume fraction in more than one spatial direction. For the case of a porous medium, we have derived a Brinkman equation with an effective viscosity different both from the effective viscosity of the effective medium and of the pure fluid.

In an earlier paper (Marchioro et al., 1999), a definition of the pressure in an incompressible mixture was developed on the basis of a covariance analysis of the averaged equations. Here we have calculated explicitly the ensemble-average pressure (Section 4) and, just as in the three-dimensional case (Tanksley and Prosperetti, 2001), we have found results in perfect agreement with the expression derived by formal means in that work. The issue is non-trivial as, in an averaged equations formulation, the mean pressure is to be found by solving the equations rather than by a closure rule.

Turning the present general results into an explicit closed-form model will require extensive numerical computations parallel to those presented in Marchioro et al. (2000, 2001) and Wang and Prosperetti (2001) for the analogous case of a suspension of spheres; this work is currently under way. In the meantime, for purposes of illustration, we have used the general formulation to develop a simple cell model which is suggestive of possible features in the structure of a complete closed formulation.

The hypothesis of Stokes flow that was made in the derivation only concerns the local flow affecting the particles. Thus, the present result for the stress tensor can also be used in situations in which the Reynolds number of the average flow is not small.

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Appendix A. The Lamb solution for disks

Following the same argument as in Lamb (1932), we decompose the solution of the Stokes equation (25) into two parts, $\mathbf{v} = \mathbf{v}^p + \mathbf{v}^h$, where both fields are divergenceless and

$$-\nabla \times \nabla \times \mathbf{v}^p = \nabla \tilde{q}, \quad -\nabla \times \nabla \times \mathbf{v}^h = 0. \tag{A.1}$$

From the second of these relations it follows that, for some function χ, $\nabla \times \mathbf{v}^h = \nabla \chi$ but, in two dimensions, $\nabla \times \mathbf{v}^h$ only has a z component and, again due to the dimensionality, χ cannot depend on z. Hence, unlike the spherical case, here we have $\chi = 0$ and $\nabla \times \mathbf{v}^h = 0$, from which we deduce that $\mathbf{v}^h = \nabla \phi$ with $\nabla^2 \phi = 0$. We write

$$\phi = v \sum_{-\infty}^{\infty} \phi_n, \tag{A.2}$$

where

$$\phi_n = s^n (\Phi_n \cos n\theta + \widetilde{\Phi}_n \sin n\theta). \tag{A.3}$$

Clearly Φ_0 can be taken to vanish without loss of generality. In writing ϕ in the form (A.2), we discard a term proportional to $\log s$ (i.e., a two-dimensional source), which is found to vanish from the boundary condition at the particle surface.

Upon taking the divergence of the first of (A.1) one finds $\nabla^2\tilde{q} = 0$, from which $\tilde{q}$ can be written in the earlier form (29). The first of (A.1) can then be solved to find

$$\mathbf{v}^p = \frac{\nu}{2a}\left\{ \sum_{\substack{-\infty \\ n\neq 0,-1}}^{\infty} \left[\frac{n+2}{2n(n+1)} s^2 \nabla_s q_n^\alpha - \frac{1}{n+1}\frac{\mathbf{r}}{a} q_n^\alpha\right] - s^2(\log s + 1)\nabla_s q_{-1}^\alpha - (2\log s + 1)\frac{\mathbf{r}}{a} q_{-1}^\alpha \right\}$$
$$+ \boldsymbol{\Omega}^\alpha \times \mathbf{r} + \frac{\nu}{2a} Q_1^\alpha s\left(1 - \frac{1}{s^2}\right)\mathbf{e}_\theta. \tag{A.4}$$

The various constants appearing in these expansions are related by the boundary condition (43) on the particle surface. One finds the following relations:

$$P_n = (n+1)P_{-n} - 4n(n+1)\Phi_{-n}, \quad \widetilde{P}_n = -(n+1)\widetilde{P}_{-n} + 4n(n+1)\widetilde{\Phi}_{-n}, \tag{A.5}$$

$$\Phi_n = (n+1)\Phi_{-n} - \frac{1}{4}\frac{n}{n-1}(1-\delta_{n1})P_{-n} + \left(\frac{a}{\nu}w_y - \frac{1}{4}P_{-1}\right)\delta_{n1} - \frac{a^2}{2\nu}\gamma_{xx}\delta_{n2}, \tag{A.6}$$

$$\widetilde{\Phi}_n = -(n+1)\widetilde{\Phi}_{-n} + \frac{1}{4}\frac{n}{n-1}(1-\delta_{n1})\widetilde{P}_{-n} + \left(\frac{a}{\nu}w_y + \frac{1}{4}\widetilde{P}_{-1}\right)\delta_{n1} - \frac{a^2}{2\nu}\gamma_{xy}\delta_{n2}. \tag{A.7}$$

From these results it easy to derive (57) while, for the stresslet (61), we find

$$a^2\nabla_r\nabla_r(s^4 q_{-2}) = 2\begin{pmatrix} P_{-2} & -\widetilde{P}_2 \\ -\widetilde{P}_{-2} & -P_{-2} \end{pmatrix}. \tag{A.8}$$

In a similar way, the multipoles appearing in Eqs. (54)–(56) can all be expressed in terms of the coefficients in the expansions (29) and (A.2).

Appendix B. Averaging

The functions $L_{\mathbf{k}}^\alpha$ (cf. Eq. (13)) which arise in the calculation are combinations of terms with the generic structure $\mathscr{F}(-i\mathbf{k})R^\alpha$, where $\mathscr{F}$ is independent of α and R^α is independent of $\mathbf{k}$. According to (8), (10), and (12), in order to calculate ensemble averages over the continuous phase, we thus need to evaluate quantities of the form

$$f_{\mathbf{k}} = \frac{1}{N!}\int d\mathscr{C}^N P(N) \frac{1}{\mathscr{V}} \sum_{\alpha=1}^{N} \exp(i\mathbf{k}\cdot\mathbf{y}^\alpha)\mathscr{F}(-i\mathbf{k})R^\alpha, \tag{B.1}$$

which can be written as

$$f_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{x}) = \mathscr{F}(\nabla)\exp(-i\mathbf{k}\cdot\mathbf{x})\frac{1}{N!}\int d\mathscr{C}^N P(N)\frac{1}{\mathscr{V}}\sum_{\alpha=1}^{N}\exp(i\mathbf{k}\cdot\mathbf{y}^\alpha)R^\alpha, \tag{B.2}$$

where $\mathscr{F}(\nabla)$ is simply obtained from $\mathscr{F}(-i\mathbf{k})$ by substituting the operator ∇ in place of $-i\mathbf{k}$. Furthermore

$$\frac{1}{N!}\int d\mathscr{C}^N P(N)\frac{1}{\mathscr{V}}\sum_{\alpha=1}^{N}\exp(i\mathbf{k}\cdot\mathbf{y}^\alpha)R^\alpha$$
$$=\frac{1}{\mathscr{V}}\int d^2x'\exp(i\mathbf{k}\cdot\mathbf{x}')\frac{1}{N!}\int d\mathscr{C}^N P(N)\sum_{\alpha=1}^{N}\delta^{(2)}(\mathbf{x}'-\mathbf{y}^\alpha)R^\alpha, \tag{B.3}$$

will be recognized as $(n\overline{R})_{\mathbf{k}}$, the $\mathbf{k}$-Fourier coefficient of $n\overline{R}$, where $\overline{R}$ is the particle average of R^α defined in (34). Thus

$$\sum_{\mathbf{k}\neq 0} f_{\mathbf{k}}\exp(-i\mathbf{k}\cdot\mathbf{x}) = \mathscr{F}(\nabla)\sum_{\mathbf{k}\neq 0}\exp(-i\mathbf{k}\cdot\mathbf{x})(n\overline{R})_{\mathbf{k}}. \tag{B.4}$$

But the summation in the right-hand side reconstructs the function $n(\mathbf{x})\overline{R}(\mathbf{x})$, except for its mean value $(n\overline{R})_0$, the Fourier coefficient corresponding to $\mathbf{k}=0$, so that, from (5), we may write

$$\beta_C(\mathbf{x})\langle f\rangle(\mathbf{x}) = f_0 + \mathscr{F}(\nabla)\left[n(\mathbf{x})\overline{R}(\mathbf{x}) - (n\overline{R})_0\right]. \tag{B.5}$$

Similarly, upon substituting (14) into (7), we find that

$$f_0 = \frac{1}{N!}\int d\mathscr{C}^N P(N)\frac{1}{\mathscr{V}}\sum_{\alpha=1}^{N} L_0^\alpha = (n\overline{L}_0)_0, \tag{B.6}$$

is just the average of $n(\mathbf{x})\overline{L}_0(\mathbf{x})$ over the cell; $(n\overline{R})_0$ is given by the same integral with R^α in place of L_0^α.

Appendix C. The operators $\mathscr{S}_l$

Consider the quantity

$$T_1(\mathbf{x}) = \int_{r\leqslant a} d^2r f(\mathbf{x}+\mathbf{r}), \tag{C.1}$$

where the function f is periodic in the unit cell and is such that the integral exists. Upon expanding f in a Fourier series one finds

$$T_1(\mathbf{x}) = v\sum_{\mathbf{k}} f_{\mathbf{k}}\exp(-i\mathbf{k}\cdot\mathbf{x})\mathscr{S}_1(-k^2a^2), \tag{C.2}$$

where, for $l = 1, 2, \ldots$,

$$\mathscr{S}_l(-z^2) = \left(\frac{2}{z}\right)^l J_l(z) = \sum_{n=0}^{\infty}\frac{1}{n!(l+n)!}\left(-\frac{1}{4}z^2\right)^n, \tag{C.3}$$

with J_l the Bessel function of the first kind. The same argument leading from (B.1) to (B.2) then shows that

$$T_1(\mathbf{x}) = v\mathscr{S}_1(f), \tag{C.4}$$

with $\mathscr{S}_1$ defined by (36). In particular, with $f = n$, this relation proves (37); furthermore,

$$\mathscr{S}_2 = \frac{1}{2} + \frac{1}{24} a^2 \nabla^2 + \cdots \tag{C.5}$$

In a similar manner it may be proven that, e.g.,

$$\int_{|\mathbf{r}| \leqslant a} \mathrm{d}^2 r \mathbf{r} f(\mathbf{x}+\mathbf{r}) = \frac{va^2}{2} \nabla \mathscr{S}_2(f), \tag{C.6}$$

$$\int_{|\mathbf{r}| \leqslant a} \mathrm{d}^2 r \mathbf{r}\mathbf{r} f(\mathbf{x}+\mathbf{r}) = \frac{av}{2} \left[\mathbf{I} \mathscr{S}_2(f) + \frac{1}{2} a^2 \nabla\nabla \mathscr{S}_3(f) \right], \tag{C.7}$$

$$\int_{r=a} \mathrm{d}S f(\mathbf{x}+\mathbf{r}) = 2\pi a \mathscr{S}_0(f), \tag{C.8}$$

$$\int_{r=a} \mathrm{d}S \mathbf{r} f(\mathbf{x}+\mathbf{r}) = va \nabla \mathscr{S}_1(f), \tag{C.9}$$

and so forth.

Appendix D. Average pressure and velocities

The calculation of the average pressure requires three separate steps: the calculation of $\langle P_\infty \rangle$, that of $\langle \tilde{q} \rangle$, and that of the contribution of the term containing $\mathbf{C}$ in (27). For the first one, according to (4) and (24), one needs to evaluate

$$\beta_\mathrm{C} \left\langle \sum_{\alpha=1}^{N} \mathbf{f}^\alpha \right\rangle = \frac{1}{N!} \int \mathrm{d}\mathscr{C}^N P(N) \sum_{\alpha=1}^{N} \mathbf{f}^\alpha - \frac{1}{N!} \int \mathrm{d}\mathscr{C}^N P(N) \chi \sum_{\alpha=1}^{N} \mathbf{f}^\alpha. \tag{D.1}$$

It is easy to show that the first term evaluates to (35), while the second one equals

$$\int_{r \leqslant a} \mathrm{d}^2 y n(\mathbf{y}) \left[\bar{\mathbf{f}}(\mathbf{y}) + (N-1) \tilde{\mathbf{f}}(\mathbf{y}) \right], \tag{D.2}$$

in which

$$\tilde{\mathbf{f}}(\mathbf{y}) = \frac{1}{(N-1)!} \int \mathrm{d}^3 y^2 \int \mathrm{d}\mathscr{C}^{N-2} P(N-1 \mid \mathbf{y}) \mathbf{f}^{(2)}(\mathbf{y}, \mathbf{y}^2, N-2), \tag{D.3}$$

where $P(N-1 \mid \mathbf{y})$ is the conditional probability for a configuration of $N-1$ particles, given that the first one occupies the position $\mathbf{y}$. With $\mathscr{F} = \bar{\mathbf{f}} + (N-1)\tilde{\mathbf{f}}$, by (C.4), we thus have

$$\beta_\mathrm{C} \mathbf{x} \cdot \left\langle \sum_{\alpha=1}^{N} \mathbf{f}^\alpha \right\rangle = \mathbf{x} \cdot [N\bar{\mathbf{f}} - \mathscr{S}_1(vn(\mathbf{x})\mathscr{F}(\mathbf{x}))]. \tag{D.4}$$

The calculation for the contribution of the term containing $\mathbf{C}$ in (27) gives

$$\frac{N}{\mathscr{V}} \left[\mathscr{S}_1(vn\mathbf{x} \cdot \mathscr{F}) - \frac{1}{2} a^2 \mathscr{S}_2 \nabla \cdot (vn\mathscr{F}) \right] = \frac{N}{\mathscr{V}} \mathbf{x} \cdot \mathscr{S}_1(vn\mathscr{F}); \tag{D.5}$$

the equality of the two members of this equation is a simple consequence of the definition (36) of the operators $\mathscr{S}_l$. The combination of (D.4) and (D.5) thus gives the last term in (32).

We now turn to the calculation of $\langle \tilde{q} \rangle$. For this quantity, with the polar axis chosen in the direction of the vector $\mathbf{k}$, the integral $L^\alpha_\mathbf{k}$ defined in (13) may be written as

$$L^\alpha_\mathbf{k} = \frac{1}{k^2}\int_0^{2\pi} \mathrm{d}\theta \left[\tilde{q}(s,\theta) z \frac{\partial}{\partial z} - \frac{\partial}{\partial s}\tilde{q}(s,\theta)\right]_{s=1} \exp(\mathrm{i}z\cos\theta), \tag{D.6}$$

where $s = r/a$, $z = ka$. Upon recalling that

$$\int_{-\pi}^{\pi} \mathrm{d}\theta \exp(\mathrm{i}z\cos\theta)\cos n\theta = 2\pi \mathrm{i}^n J_n(z), \qquad \int_{-\pi}^{\pi} \mathrm{d}\theta \exp(\mathrm{i}z\cos\theta)\sin n\theta = 0, \tag{D.7}$$

one finds

$$L^\alpha_\mathbf{k} = \frac{2\pi v}{z}\left[-J_1(z)P^\alpha_0 + \sum_{n=1}^{\infty} \mathrm{i}^n \left(\frac{2n}{z} J_n(z) P^\alpha_{-n} - J_{n+1}(z)(P^\alpha_n + P^\alpha_{-n})\right)\right]. \tag{D.8}$$

But it is easy to verify from (29) and (30) that

$$P^\alpha_l = \frac{a^l}{l!}\partial^l_x q^\alpha_l = \frac{(\mathrm{i}a^2)^l}{l! z^l}(-\mathrm{i}\mathbf{k}\cdot\nabla)^l q^\alpha_l, \tag{D.9}$$

$$P^\alpha_{-l} = \frac{a^l}{l!}\partial^l_x (s^{2l} q^\alpha_{-l}) = \frac{(\mathrm{i}a^2)^l}{l! z^l}(-\mathrm{i}\mathbf{k}\cdot\nabla)^l (s^{2l} q^\alpha_{-l}), \tag{D.10}$$

so that (D.8) can be written in the coordinate-free form

$$L^\alpha_\mathbf{k} = \pi v \left\{ -S_1 P^\alpha_0 + \sum_{l=1}^{\infty} \frac{1}{l!}\left(-\frac{a^2}{2}\right)^l (-\mathrm{i}\mathbf{k}\cdot)^l \left[\frac{4l}{z^2} S_l \nabla^{(l)}_r (s^{2l} q^\alpha_{-l}) - S_{l+1}\nabla^{(l)}_r (q^\alpha_l + s^{2l} q^\alpha_{-l})\right]\right\}, \tag{D.11}$$

where $S_l(-z^2)$ is given by (C.3). The relevant solution of (16) near the generic particle is readily found and the result for L^α_0 is

$$L^\alpha_0 = -\pi v P^\alpha_0. \tag{D.12}$$

Upon using (B.5) and (B.6), we then find

$$\begin{aligned}\beta_C \mu \langle \tilde{q} \rangle = {} & -\frac{\mu v}{a^2}[(nv\bar{q}_0)_0 + \mathscr{S}_1(nv\bar{q}_0 - (nv\bar{q}_0)_0)] + \upsilon v \mu \sum_{l=1}^{\infty}\frac{1}{l!}\left(-\frac{a^2}{2}\right)^l \mathscr{S}_{l+1}(\nabla\cdot)^{(l)} \\ & \times \left[n\overline{\nabla^{(l)}_r (q^\alpha_l + s^{2l} q_{-l})}\right] + 2\pi v (\nabla^2)^{-1} \sum_{l=1}^{\infty} \frac{1}{(l-1)!}\left(-\frac{a^2}{2}\right)^{l-1} \mathscr{S}_l (\nabla\cdot)^{(l)} \left[n\overline{\nabla^{(l)}_r (s^{2l} q_{-l})}\right].\end{aligned} \tag{D.13}$$

In writing this relation we have dropped the constant terms analogous to $(n\overline{R})_0$ in (B.5). Indeed, in the first summation, all terms are differentiated at least once. As for the second summation, we note that from (C.4), for example,

$$\mathscr{S}_1\nabla\cdot\left(n\overline{\nabla_r(s^2\tilde{q}_{-1})}\right)=\frac{1}{v}\int_{r=a}\mathrm{d}S\mathbf{r}\cdot\overline{\nabla_r(s^2\tilde{q}_{-1})}, \tag{D.14}$$

from which it is evident that the constant part of $\overline{\nabla_r(s^2\tilde{q}_{-1})}$ (which is what would be removed by subtracting the constant term) gives no contribution to the integral anyway; an analogous argument is applicable to the other terms. Finally, from the definition (31), the operator $\mathscr{S}_1$ acting on a constant equals 1 so that the first and third terms in the right-hand side cancel and the result (32) given in the text follows.

Turning now to the disperse-phase velocity, we recall the fact, noted in Section 5, that $\mathbf{u}_\mathrm{D}-\mathbf{U}_\infty$ is periodic and harmonic and, therefore, its average can be calculated as in (12) and (13). One readily finds

$$\mathbf{L}_k^\alpha=\frac{2v}{z}\left(J_1(z)\mathbf{w}^\alpha+\mathrm{i}a^2\frac{J_2(z)}{z}\boldsymbol{\Omega}\times\mathbf{k}+\mathrm{i}\frac{J_2(z)}{z}a^2\boldsymbol{\gamma}\cdot\mathbf{k}\right), \tag{D.15}$$

$$\mathbf{L}_0^\alpha=v\mathbf{w}^\alpha, \tag{D.16}$$

from which (49) follows directly via (B.5).

For the continuous-phase velocity one has

$$\mathbf{L}_k^\alpha=-\frac{2v}{z}\left(J_1(z)\mathbf{w}^\alpha+\mathrm{i}a^2\frac{J_2(z)}{z}\boldsymbol{\Omega}^\alpha\times\mathbf{k}+\mathrm{i}\frac{J_2(z)}{z}a^2\boldsymbol{\gamma}\cdot\mathbf{k}\right)-\frac{2v\nu\mathrm{i}}{az^2}J_1(z)Q_1^\alpha+\mathbf{W}^\alpha, \tag{D.17}$$

where $\mathbf{W}^\alpha$ is a vector whose x-component (along $\mathbf{k}$) vanishes, while the y component equals:

$$W_y^\alpha=\frac{2\pi\nu a}{z^2}\sum_{l=1}^{\infty}\mathrm{i}^{l-1}l\left[4(l+1)J_{l+1}(z)\widetilde{\Phi}_{-l}^\alpha-\left(J_{l+1}(z)+\frac{2}{z}J_l(z)\right)\widetilde{P}_{-l}^\alpha\right]. \tag{D.18}$$

We may account for this structure by introducing vectors $\widetilde{\boldsymbol{\Phi}}_{-l}^\alpha=(0,\widetilde{\Phi}_{-l}^\alpha)$ and $\widetilde{\mathbf{P}}_{-l}^\alpha=(0,\widetilde{P}_{-l}^\alpha)$ and noting that they may be expressed as

$$\widetilde{\boldsymbol{\Phi}}_{-l}=-\frac{(\mathrm{i}a^2)^{l-1}a}{l!}\frac{(-\mathrm{i}\mathbf{k}\cdot\nabla)^{l-1}}{z^{l-1}}\left(\mathbf{I}-\frac{\mathbf{kk}}{k^2}\right)\cdot\nabla_r(s^{2l}\phi_{-l}), \tag{D.19}$$

where $\mathbf{I}$ is the identity tensor. In a similar fashion

$$\widetilde{\mathbf{P}}_{-l}=-\frac{(\mathrm{i}a^2)^{l-1}a}{l!}\frac{(-\mathrm{i}\mathbf{k}\cdot\nabla)^{l-1}}{z^{l-1}}\left(\mathbf{I}-\frac{\mathbf{kk}}{k^2}\right)\cdot\nabla_r(s^{2l}p_{-l}). \tag{D.20}$$

With this step, proceeding as before, we recover the result (52) given in the text.

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